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Modified Factorization Method and Supersymmetric Quantum Mechanics

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We suggest a modified factorization scheme within a supersymmetric framework which affords a consistent treatment of a wide class of Schrödinger potentials. A consequence of this is the possibility of deriving a boson–fermion Hamiltonian with linear interaction.

There has been growing interest in the search for supersymmetry in quantum mechanical problems in the past few years [see Lahiri *et al.* (1990) for a recent survey on the subject]. In most cases, supersymmetry has been formulated for a class of potentials which are shape-invariant (Gendenshtein, 1983) or constructed (Levai, 1989; Chuan, 1990) to be so. As is well known, the essence of supersymmetry rests in the possibility of factorizing (Stahlhofen and Bleuler, 1989) the Schrödinger Hamiltonian and thereby inducing a governing (Cervaro, 1991) superpotential. In effect this means solving a nonlinear equation which belongs to the Riccati class.

In the present study we introduce a modified (Leach, 1985) version of the so-called factorization scheme in the framework of supersymmetric quantum mechanics. As shall be seen, we are able to account for a whole sequence of potentials ranging from a polynomial to other varieties, including the singular ones (Casahorran and Nam, 1991). An interesting consequence of our scenario is the possibility of touching upon supersymmetric Hamiltonians possessing (Stedman, 1985; Jarvis and Stedman, 1984) boson–fermion interaction terms.

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To embark upon our scheme, we consider the time-independent Schrödinger equation in one dimension (in units $\hbar = 2m = 1$):

$$L\psi = \left[-\frac{d^2}{dx^2} + (V - E_0) \right] \psi = 0 \tag{1}$$

under the influence of a potential $V = V(x)$ for $E = E_0$. Left multiplying the operator equation (1) by some function $f(x)$, we observe that we can always express

$$fL = \left(-\frac{d}{dx} + \alpha \right)\left(f\frac{d}{dx} + \beta \right) \tag{2}$$

provided we restrict

$$\beta = \alpha f - f' \tag{3a}$$

$$V - E_0 = (\alpha - f'/f)^2 - (\alpha - f'/f)' \tag{3b}$$

This puts us in a position to establish contact with the supersymmetric theory. We note that the superpotential $W(x)$ also satisfies a relation similar to (3b):

$$V - E_0 = W^2 - W' \tag{4}$$

This suggests an identification

$$W = \alpha - f'/f \tag{5}$$

which must hold good for any arbitrary f.

Now replacing α by the superpotential itself, we deduce a modified superpotential,

$$W_{\text{mod}} = W - f'/f \tag{6}$$

As a result the $N = 2$ supersymmetric charges assume the form

$$Q = (d/dx + W - f'/f)c^+ \tag{7a}$$

$$Q^\dagger = (-d/dx + W - f'/f)c \tag{7b}$$

where c and c^+ are the fermionic annihilation and creation operators, respectively. The charges Q and $Q^\dagger$ as usual satisfy the $N = 2$ supersymmetric algebra:

$$Q^2 = Q^{+2} = 0 \tag{8a}$$

$$H_s = \{Q, Q^+\}, \qquad [H_s, Q] = [H_s, Q^+] = 0 \tag{8b}$$

For W_{mod} we can express the ground-state wavefunction ϕ_0 in terms of the ψ_0 corresponding to W:

$$\phi_0 = \exp\left[-\int W_{mod}\, dy\right] = f\psi_0 \tag{9}$$

The above formalism can be applied to a wide class of potentials. For example, if we choose $f=$const along with $W=x$, we see immediately that equation (6) reduces to the case of the harmonic oscillator.

Consider the less trivial possibility $f'/f=\lambda x^2$ [which implies $f=\exp(\lambda x^3/3)$, where λ is a parameter] for the same choice of W, namely $W=x$. We read off from (7)

$$\begin{aligned} Q &= [b-\lambda(b+b^+)^2]c^+ \\ Q^\dagger &= [b^+-\lambda(b+b^+)^2]c \end{aligned} \tag{10}$$

where b and b^+ are the bosonic annihilation and creation operators, that is, $[b, b^+]=1$.

The set (10) generates the supersymmetric Hamiltonian

$$H_s(\text{int}) = c^+c + b^+b - 4\lambda c^+ca + 2\lambda a - \lambda a^3 + \lambda^2 a^4 \tag{11}$$

where $a=b+b^+$ and $B=b-\lambda a^2$. In terms of B, $H_s(\text{int})=c^+c[B, B^+]+B^+B$. But B is not a bosonic operator: $[B, B^+]=1-4\lambda a\neq 1$.

It is noteworthy that $H_s(\text{int})$ has picked up a nontrivial interaction between a boson and a fermion variable and remarkably coincides with the form previously proposed by Jarvis and Stedman (1984). Treating λ as a coupling parameter, it is transparent from (11) that cubic anharmonicity is present at the same order (λ) as the fermion–boson coupling. As emphasized by Stedman (1985), this is essential, since at zeroth order in λ these terms mix under $Q=(b-\lambda a^2)c^+$. One can show in perturbation theory that to second order in λ the ground state is unshifted. Jarvis and Stedman (1984) consider applications to supersymmetric Jahn–Teller systems. More recently, thermodynamic implications of $H_s(\text{int})$ have been studied (Steeb *et al.*, 1989).

Using the representations of c and c^+, one can express equation (11) as a 2×2 diagonal matrix with entries

$$H^s_\pm(\text{int}) = \frac{1}{2}\left(-\frac{d^2}{dx^2}+x^2\right) \mp \frac{1}{2} - 2\sqrt{2}\,\lambda\,(x^3 \mp x) + 4\lambda^2 x^4$$

We notice that in the partner Hamiltonians, cubic as well as quartic anharmonic terms are present to $O(\lambda)$ and $O(\lambda^2)$, respectively. The Schrödinger potential corresponding to this system is of the polynomial form $V(x)=ax+bx^2+cx^3+dx^4$, where a, b, c, and d are constants. Extensive literature exists on the study of such a potential, see Kaushal (1991) for a nice treatment. In this way other variants of H_s(int) may be proposed by suitably modifying the functions f and W.

We now turn to some singular superpotentials. Assuming that singularities exist at the points $x=\pm c$, we choose $f(x)=x^2-c$ along with $W(x)=x^3$. We find

$$W_{\text{mod}}=x^3-\frac{2x}{x^2-c} \tag{12}$$

yielding

$$V_+=x^6-7x^2-\frac{4c^2-2}{x^2-c}-4c^2 \tag{13a}$$

$$V_-=x^6-x^2-\frac{4c^2+2}{x^2-c}+\frac{8x^2}{(x^2-c)^2}-4c^2 \tag{13b}$$

The Schrödinger potential is $V(x)=x^6-7x^2-(4c^2-2)/(x^2-c)$ and $E_0=4c$. Roy and Varshni (1991) have recently studied the pair (V_+, V_-) in the context of exploring solvability in quasiexact systems as defined by Turbiner and Ushveridze (1987). Roy and Varshni (1991) found the two exact solutions for $V(x)$ to be

$$E_0=-2\sqrt{2}, \qquad \psi_0\sim(x^2+1/\sqrt{2})\exp(-x^4/4) \tag{14a}$$

$$E_2=2\sqrt{2}, \qquad \psi_2\sim(x^2-1/\sqrt{2})\exp(-x^4/4) \tag{14b}$$

where (0) and (2) denote the order of the levels.

A particular case of W_{mod} above occurs when $c=0$. This was considered by Cooper and Freedman (1983) and gives $V_+=x^6-5x^2$, $V_-=x^6+x^2+2/x^2$. Here also the Schrödinger potential $V(x)=x^6-5x^2$ belongs to the family (Turbiner and Ushveridze, 1987) of quasiexactly solvable potentials $V(x)=x^6-(8_j+3)x^2$ for $j=\frac{1}{4}$. Note that V_- is singular when $x=0$.

Finally, we remark on the nonpolynomial potential (Filho and Ricotta, 1989)

$$V=x^2+\frac{\lambda x^2}{1+gx^2}, \qquad \lambda=\lambda(g) \tag{15}$$

which has some applicability in quantum optics. This V can be accommodated in our modified factorization scheme by setting

$$W=x-\frac{2gx}{1+gx^2} \tag{16}$$

and choosing

$$f(x)=\mu(x^2-\lambda_0)(x^2-\lambda_1)\cdots(x^2-\lambda_n) \tag{17}$$

From (16) and (17) we find

$$W_{\text{mod}}=x-\frac{2gx}{1+gx^2}+2x\sum_{i=0}^{n}\frac{1}{x^2-\alpha_i} \tag{18}$$

See Roy *et al.* (1991) for a detailed analysis of the potential V in (15).

To conclude, we have proposed a modified factorization scheme within a supersymmetric framework which encompasses a wide variety of potentials. We have explored some forms, but we believe that our model can be successfully employed to deal with more complicated (Beckers and Debergh, 1991; deLaney and Nieto, 1990) types of potentials. We shall take this up in more detail elsewhere. A word about the function f introduced in our treatment: Since, according to equation (9), f is directly related to the ground-state wave function, the presence of a singularity in it will inevitably lead to problems with the normalizability and square integrability of ϕ_0. This is overcome if f is taken to be, say, a polynomial or an exponential function, as has been done in the cases we have considered.

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